

Dynatomic cycles for morphisms of projective varieties

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Abstract. We prove the effectivity of the zero-cycles of formal periodic points, dynatomic cycles, for morphisms of projective varieties. We then analyze the degrees of the dynatomic cycles and multiplicities of formal periodic points and apply these results to the existence of periodic points with arbitrarily large primitive periods.

1. INTRODUCTION

Let K be a field and X/K a projective variety. Let $\phi : X/K \rightarrow X/K$ be a morphism defined over K . We can iterate the morphism ϕ and study the properties of the periodic points of the resulting dynamical system. In this paper, we consider K an algebraically closed field and study the zero cycle of formal n -periodic points, the n -th *dynatomic cycle*, and show that it is effective for all positive integers n (Theorem 2.20), resolving a conjecture of Morton and Silverman in the affirmative [13, Conjecture 1.1]. We further show that the periodic points of formal period $n \neq 0$ in K with multiplicity one have primitive period n (Theorem 3.1). We relate the degrees of the dynatomic cycles to periodic Lefschetz numbers and use information about the degrees to investigate the existence of primitive periodic points. In particular, we show that the dynamical systems constructed on Wehler K3 surfaces and the dynamical systems arising from morphisms of projective space have periodic points with arbitrarily large primitive periods (Theorem 4.12 and Theorem 4.18). Much of this work is from the author's doctoral thesis [6, Chapter 3].

We now describe our results in more detail. We denote ϕ^n as the n -th iterate of the morphism ϕ . If $\phi^n(P) = P$ for some $P \in X(K)$ and $n \in \mathbb{N}$, then P is called a *periodic point of period n* for ϕ . If n is the smallest such period, then n is called the *primitive period* of P . Consider the cycles in $X \times X$: The graph of ϕ^n defined as $\Gamma_n = \sum_{x \in X} (x, \phi^n(x))$ and the diagonal $\Delta = \sum_{x \in X} (x, x)$.

Definition 1.1. For $n \geq 1$, we say that ϕ^n is *non-degenerate* if Δ and Γ_n intersect properly.

Remark. If ϕ^n is non-degenerate, then ϕ^d is non-degenerate for all $d \mid n$.

Assume that ϕ^n is non-degenerate, and let $P \in X(K)$. Define $a_P(n)$ to be the intersection multiplicity of Γ_n and Δ at (P, P) and

$$\Phi_n(\phi) = \sum_{P \in X} i(\Gamma_n, \Delta; P)(P) = \sum_{P \in X} a_P(n)(P).$$

Notice that this intersection contains all of the periodic points of period n for ϕ . We want to examine only the primitive n -periodic points. Define

$$a_P^*(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(n)$$

and

$$\Phi_n^*(\phi) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \Phi_d(\phi) = \sum_{P \in X} a_P^*(n)(P),$$

where μ is the Möbius function.

Definition 1.2. We call $\Phi_n^*(\phi)$ the n -th *dynatomic cycle* and $a_P^*(n)$ the *multiplicity* of P in $\Phi_n^*(\phi)$. If $a_P^*(n) > 0$, then we call P a *formal periodic point of formal period n* .

All periodic points of primitive period n are points of formal period n (Proposition 4.1(2)), but there may be periodic points with formal period n and primitive period strictly less than n [12, Theorem 2.4].

For $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^1$, a single variable polynomial map, Morton showed that $\Phi_n^*(\phi)$ is effective; and in the case where $n \neq 0$ in K and it has points with multiplicity greater than one, its points of nonzero multiplicity are exactly the points of primitive period n [12, Theorem 2.4 and Theorem 2.5]. Morton and Silverman went on to show the effectivity of $\Phi_n^*(\phi)$ for a non-degenerate morphism of a nonsingular projective curve and for a non-degenerate automorphism of projective space [13, Theorem 2.1 and Theorem 3.1]. They also conjectured effectivity for non-degenerate morphisms of nonsingular projective varieties [13, Conjecture 1.1].

In Section 2 we prove that $\Phi_n^*(\phi)$ is effective for non-degenerate morphisms of non-singular, irreducible, projective varieties and describe the possible values of n for which a periodic point P of ϕ has non-zero multiplicity in $\Phi_n^*(\phi)$ (Theorem 2.20).

As in the one-dimensional case, the proof is carried out by carefully examining when the multiplicity of a fixed point P in $\Phi_n(\phi)$ is greater than the multiplicity of P in $\Phi_1(\phi)$. However, several new ideas and a lot of additional work are needed in the higher dimensional case. Some of the difficulties encountered are taking into account the higher Tor modules in the intersection theory, which turn out to all be identically 0 (Theorem 2.3), using the theory of standard bases to obtain information about the multiplicity of a point in $\Phi_n(\phi)$ (Proposition 2.17), and iterating local power series representations of the morphism.

From this detailed analysis of the multiplicities, in Section 3 we show that periodic points of formal period n with multiplicity one and $n \neq 0$ in K have primitive period n . In other words, $a_P^*(n) = 1$ for $n \neq 0$ in K implies that P is a periodic point of primitive period n . This generalizes [12, Theorem 2.5] to morphisms of projective varieties.

In Section 4.1 we state some basic properties of $\Phi_n(\phi)$ and $\Phi_n^*(\phi)$. In Section 4.2 we note the similarity to periodic Lefschetz numbers, and in Section 4.3 we state results similar to those of [5, 8, 17] on the existence of primitive periodic points. In particular, if P is a periodic point, then the sequence $a_P(n)$ for $n \neq 0$ in K is bounded (Theorem 4.10), and if $\deg(\Phi_n)$ is unbounded for $n \neq 0$ in K , then there are periodic points with arbitrarily large primitive periods and infinitely many periodic points (Corollary 4.11). In Section 4.4 these results are applied to dynamical systems on Wehler K3 surfaces studied in [2, 18] and in Section 4.5 to dynamical systems arising from morphisms of projective space.

The cycles $\Phi_n(\phi)$ and $\Phi_n^*(\phi)$ occur with great frequency in the literature, under a variety of notations and with a number of results stemming from the fact they are effective, see for example [9, 10, 11, 12, 13, 14, 15, 19]. In particular, [12, 19] contain Galois theoretic results in the single-variable polynomial case where $\Phi_n^*(\phi)$ has no points of multiplicity greater than one; many of the arguments of these two articles carry through to the higher dimensional case given that $\Phi_n^*(\phi)$ is effective (see [6, Chapter 4]).

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2. EFFECTIVITY OF $\Phi_n^*(\phi)$

Let K be an algebraically closed field and let X be a non-singular, irreducible, projective variety of dimension b defined over K . Let $\phi : X \rightarrow X$ be a morphism defined over K such that ϕ^n is non-degenerate. Define the cycles in $X \times X$: The graph of ϕ^n defined as $\Gamma_n = \sum_{x \in X} (x, \phi^n(x))$ and the diagonal $\Delta = \sum_{x \in X} (x, x)$. Let R_P be the local ring of $X \times X$ at (P, P) and let $I_\Delta, I_{\Gamma_n} \subset R_P$ be the ideals of Δ and Γ_n , respectively. The following steps outline the proof of the effectivity of $\Phi_n^*(\phi)$.

- (1) Define the intersection multiplicity and show that $\Phi_n^*(\phi)$ is a zero-cycle.
- (2) Show that the naive intersection theory is, in fact, correct (Theorem 2.3). Specifically, show that

$$\mathrm{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0 \quad \text{for all } i > 0.$$

- (3) Determine conditions on n for when $a_P(n) > a_P(1)$ (Proposition 2.17).

(4) Show that $a_P^*(n) \geq 0$ for all P and n (Theorem 2.20).

In what follows, the concept of dimension will be used in several different contexts. We will denote

- $\dim R$ for the Krull dimension of a ring R ,
- $\dim M$ for the Krull dimension of $R/\text{Ann}(M)$ where $\text{Ann}(M)$ is the annihilator of the R -module M , and
- $\dim_K V$ for the dimension of the finite dimensional K -vector space V .

2.1. Intersection multiplicity. Using Serre's definition of intersection multiplicity and working over the completion \widehat{R}_P of R_P , we have

$$i(\Delta, \Gamma_n; P) = \sum_{i=0}^{b-1} (-1)^i \dim_K(\text{Tor}_i(\widehat{R}_P/I_\Delta, \widehat{R}_P/I_{\Gamma_n})).$$

Since ϕ^n is non-degenerate, the cycles Δ and Γ_n intersect properly. We also know $X \times X$ has dimension $2b$, Δ has dimension b , and Γ_n has dimension b . Consequently, $\Phi_n(\phi)$ is a zero-cycle with a finite number of points with non-zero multiplicity. Therefore, $\Phi_n^*(\phi)$ is also a zero-cycle.

In local coordinates, we have

$$\widehat{R}_P \cong K[[x_1, \dots, x_b, y_1, \dots, y_b]].$$

Definition 2.1. Let $\phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \dots, \phi_b(\mathbf{x})]$, where $\mathbf{x} = (x_1, \dots, x_b)$. Then denote

$$\phi^n(\mathbf{x}) = [\phi_1^{(n)}(\mathbf{x}), \dots, \phi_b^{(n)}(\mathbf{x})]$$

as the coordinates of the n -th iterate of ϕ .

Then we have

$$I_\Delta = (x_1 - y_1, \dots, x_b - y_b) \quad \text{and} \quad I_{\Gamma_n} = (\phi_1^{(n)}(\mathbf{x}) - y_1, \dots, \phi_b^{(n)}(\mathbf{x}) - y_b).$$

We will use the non-degeneracy of ϕ^n and the following theorem to show that $\text{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0$ for all $i > 0$.

Theorem 2.2. ([16, Corollary to Theorem V.B.4]) *Let (R, \mathfrak{m}) be a regular local ring of dimension b , and let M and N be two non-zero finitely generated R -modules such that $M \otimes N$ is of finite length. Then $\text{Tor}_i(M, N) = 0$ for all $i > 0$ if and only if M and N are Cohen-Macaulay modules and $\dim M + \dim N = b$.*

Theorem 2.3. *Let X be a non-singular, irreducible, projective variety defined over a field K and $\phi : X \rightarrow X$ a morphism defined over K such that ϕ^n is non-degenerate. Let $P \in X(K)$ and R_P the local ring of $X \times X$ at (P, P) . Let $\Delta, \Gamma_n \subset X \times X$ be the diagonal and the graph of ϕ^n , respectively, and let $I_\Delta, I_{\Gamma_n} \subset R_P$ be their ideals. Then, $\text{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0$ for all $i > 0$.*

Proof. Let $b = \dim X$, then we have $\dim X \times X = 2b$ and $\dim \Delta = \dim \Gamma_n = b$. The ideals I_Δ and I_{Γ_n} are each generated by b elements and Δ and Γ_n intersect properly. Therefore,

$$\dim_K(R_P/(I_\Delta + I_{\Gamma_n})) = \text{length}(R_P/I_\Delta \otimes R_P/I_{\Gamma_n}) < \infty.$$

By [16, Proposition III.B.6] the union of the generators of I_Δ and the generators of I_{Γ_n} are a system of parameters for R_P . Because R_P is Cohen-Macaulay by [16, Corollary 3 to Theorem IV.D.9] we can apply [16, Corollary to Theorem IV.B.2] to I_Δ and its generators to conclude that R_P/I_Δ is Cohen-Macaulay of dimension b and, similarly with I_{Γ_n} , to conclude that R_P/I_{Γ_n} is Cohen-Macaulay of dimension b .

We have fulfilled the hypotheses of Theorem 2.2; consequently, we have that

$$\text{Tor}_i(R_P/I_\Delta, R_P/I_{\Gamma_n}) = 0 \quad \text{for all } i > 0.$$

□

2.2. Tor₀ module. If P is not a periodic point, then $a_P(n) = 0$ for all n , so we will assume that P is a periodic point. If $a_P(1) = 0$, then P has some primitive period $m > 1$. If $m \nmid n$ then $a_P(n) = 0$, so we may replace ϕ by ϕ^m and assume that P is a fixed point for ϕ and, hence, $a_P(1) > 0$. For P , a fixed point of ϕ , we can iterate a local representation of ϕ as a family of power series.

From Theorem 2.3 we know the naive intersection index

$$\begin{aligned} a_P(n) &= i(\Delta, \Gamma_n; P) \\ &= \dim_K(\text{Tor}_0(R_P/I_\Delta, R_P/I_{\Gamma_n})) \\ &= \dim_K(R_P/(I_\Delta + I_{\Gamma_n})) \end{aligned}$$

is, in fact, correct in our situation. To prove the effectivity of $\Phi_n^*(\phi)$, we will use conditions on n for $a_P(n)$ to be greater than $a_P(1)$. To determine these conditions, we will consider local power series representations of ϕ and the theory of standard bases. For information on standard bases, see [3, Chapter 4]. Below, we recall the needed terminology.

Definition 2.4. Recall that for a ring of formal power series $K[[X_1, \dots, X_h]]$, we can write an element $f \in K[[X_1, \dots, X_h]]$ as

$$f = \sum_{v \in \mathbb{N}^h} f_v X^v.$$

The monomial *support* of f is defined as

$$\text{supp}(f) = \{f_v X^v \mid f_v \neq 0\}.$$

If $f \neq 0$, then $\text{supp}(f)$ has a least element under any admissible monomial ordering. We call this least element the *leading monomial* of f , denoted by $LM(f)$. We denote $v(f)$ the exponent of the leading monomial. Then

$$LM(f) = f_{v(f)} X^{v(f)}$$

and we call $X^{v(f)}$ the *leading term* of f and denote it by $LT(f)$.

Let I be an ideal in $K[[X_1, \dots, X_h]]$. We define the *leading term ideal* of I as

$$LT(I) = \text{the polynomial ideal generated by } \{X^v \mid \exists f \in I \text{ with } LT(f) = X^v\}.$$

Definition 2.5. A non-zero element $f \in K[[X_1, \dots, X_h]]$ is called *self-reduced* with respect to an admissible monomial ordering if

$$LT(f) \nmid F \text{ for all } F \in \text{supp}(f) - LT(f).$$

Finally we recall three facts that we will need (see [3, Chapter 4.4]).

Theorem 2.6. *The following are equivalent.*

- (1) *There exists a standard basis for I .*
- (2) *Every $f \in K[[X_1, \dots, X_h]]$ has a unique standard remainder modulo I .*
- (3) *Every $f \in K[[X_1, \dots, X_h]]$ has a standard remainder modulo I .*

Theorem 2.7. *Every ideal $I \subset K[[X_1, \dots, X_h]]$ has a universal standard basis.*

Theorem 2.8. *Let $I \subset K[[X_1, \dots, X_h]]$ be an ideal with $\dim K[[X_1, \dots, X_h]]/I = 0$. Then $K[[X_1, \dots, X_h]]/I$ is isomorphic as a K -vector space to $\text{Span}(X^v \mid X^v \notin LT(I))$.*

For the most part, we will not be concerned with the particular admissible ordering that is used, so it what follows we fix an admissible monomial ordering. When necessary, we will specify a particular ordering.

Remark. For notation convenience, define $I_n = I_\Delta + I_{\Gamma_n}$.

Corollary 2.9. Consider the ideal $I_n = (I_\Delta + I_{\Gamma_n}) \subset \widehat{R_P}$. Then

$$a_P(n) = \dim_K(\widehat{R_P}/I_n) = \dim_K(\text{Span}(X^v \mid X^v \notin LT(I_n))).$$

Proof. Apply Theorem 2.8 to $\widehat{R_P}$ and I_n . □

Lemma 2.10. Assume ϕ^n is non-degenerate. Then $a_P(n) \geq a_P(1)$ for all $n \in \mathbb{N}$.

Proof. It is clear that

$$\Gamma_1 \cap \Delta \subseteq \Gamma_n \cap \Delta$$

and we have a local representation of $\phi = (\phi_1, \dots, \phi_n)$ at the fixed point P . Iterating this representation involves taking combination of the ϕ_i and hence are all elements of the original ideal I_{Γ_1} . Hence, we have

$$I_{\Gamma_n} + I_\Delta = I_n \subseteq I_1 = I_{\Gamma_1} + I_\Delta.$$

Therefore,

$$LT(I_n) \subseteq LT(I_1)$$

which implies $a_P(n) \geq a_P(1)$. □

For a fixed point P , if $a_P(n) > a_P(1)$, then some monomial of a power series representation of ϕ near P does not appear in the n -th iterate of that representation. So we need to examine when monomials may have coefficient 0 after iteration. We next show that we may reduce to the case where the generators of the ideal are self-reduced.

Lemma 2.11. Let $I \subset K[[X_1, \dots, X_h]]$ be an ideal generated by $\{f_1, \dots, f_m\}$ with $\dim K[[X_1, \dots, X_h]]/I = 0$. Let $u_i \in K[[X_1, \dots, X_h]]$ be a unit such that $u_i f_i$ is self-reduced for each $1 \leq i \leq h$ and define $uI = (u_1 f_1, \dots, u_m f_m)$. Then

$$\dim_K(\text{Span}(X^v \mid X^v \notin LT(I))) = \dim_K(\text{Span}(X^v \mid X^v \notin LT(uI))).$$

Remark. By [1, Corollary 2.2] applied to

$$LT(I) = LT((f_1, \dots, f_m)),$$

we know each there exist units $u_i \in K[[X_1, \dots, X_h]]$ such that each $u_i f_i$ is self-reduced.

Proof. Since each u_i is a unit, we have $v(LT(u_i)) = 0$ and $LT(u_i f_i) = LT(f_i)$ (and similarly for any combinations of the f_i). Hence we have

$$LT(I) = LT((f_1, \dots, f_m)) = LT((u_1 f_1, \dots, u_m f_m)).$$

□

We now show that we can also exclude from consideration those monomials that are products of other monomials in the monomial support of ϕ under iteration.

Example 2.12. We have

$$\begin{aligned} \phi_1(x, y, z) &= x^4 + \boxed{x^2 z^2} + xy \\ \phi_2(x, y, z) &= y^4 + xz^2 \\ \phi_3(x, y, z) &= z^4. \end{aligned}$$

We have that xy creates an additional $x^2 z^2$ term under iteration.

Lemma 2.13. Assume that $H \in \text{supp}(\phi_i)$ is a monomial which is a product of other monomials under iteration. Then $LT(\phi_1, \dots, \phi_d) = LT(\phi_1, \dots, \phi_i - H, \dots, \phi_d)$.

Proof. Assume that $H \in \text{supp}(\phi_i)$ and $H \notin \text{supp}(\phi_i^{(n)})$ for some n and that there is no other monomial in $\text{supp}(\phi)$ that not in $\text{supp}(\phi^{(n)})$. Then notice that for some $m > n$ we will also have $H \in \text{supp}(\phi_i^{(m)})$ since H is the product of other least monomials under iteration. If it were true that $a_P(n) > a_P(1)$, then it would also be true that $a_P(m) < a_P(n)$ which contradicts Lemma 2.10. \square

Definition 2.14. Let $f \in K[[X_1, \dots, X_h]]$ be a non-zero element. Then we call $F \in \text{supp}(f)$ a *least monomial* of f if

$$\{X^v \mid X^v \in \text{supp}(f), v \neq 0, \text{ and } X^v \text{ divides } F\} = \{F\}$$

and F is not the product of other least monomials in the support of ϕ under iteration.

It is clear that one of the monomials in the monomial support of ϕ^n can be 0 in K when the coefficient λ_i of the linear term x_i in $\text{supp}(\phi_i)$ satisfies $\lambda_i \neq 1$ and $\lambda_i^n = 1$ for some $1 \leq i \leq b$. It is also possible to have a coefficient of 0 after iteration when $\text{char } K = p$ and $n = Mp^e$ for some $e \geq 1$. Lemma 2.16 gives general conditions for when the coefficient of a least monomial is divisible by p after iteration.

Denote $d\phi_P$ as the map induced by ϕ on the cotangent space of X at P . Recall that we are assuming that P is a fixed point of ϕ and that K is algebraically closed. Therefore, $d\phi_P$ is a $b \times b$ matrix and can always be put in Jordan-canonical form, with Jordan blocks J_1, \dots, J_k of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_i \end{pmatrix}.$$

Having a non-trivial Jordan block causes more complicated interaction between the ϕ_i through the additional linear terms. Consider as an example J_1 of size \mathfrak{v} . We have

$$\begin{aligned} \phi_1 &= \lambda_1 x_1 + x_2 + \text{higher order terms} \\ \phi_2 &= \lambda_1 x_2 + x_3 + \text{higher order terms} \\ &\vdots \\ \phi_{\mathfrak{v}} &= \lambda_1 x_{\mathfrak{v}} + \text{higher order terms.} \end{aligned}$$

Along with the linear terms $\lambda_1 x_1, \dots, \lambda_1 x_{\mathfrak{v}}$, we also have the linear terms $x_2, \dots, x_{\mathfrak{v}}$. Notice that for F , a least monomial in the monomial support of ϕ_i in a non-trivial Jordan block of $d\phi_P$, it may be that $F \in \text{supp}(\phi_j)$ for some other ϕ_j in the same Jordan block of $d\phi_P$. We will be concerned with $F \notin \text{supp}(\phi_j^{(n)})$ for every ϕ_j in the same Jordan block of $d\phi_P$. First we describe the coefficients of a least monomial under iteration.

Lemma 2.15. Let $F = \prod x_i^{e_i}$ and assume the monomial support of ϕ_{i_t} with $1 \leq i_t \leq b$ contains a least monomial that is a constant multiple of F . Assume that ϕ_{i_t} is in a Jordan block of $d\phi_P$ of size $\mathfrak{v} \geq 1$ with eigenvalue λ . Label the rows $1, \dots, \mathfrak{v}$ corresponding to $\phi_{i_1}, \dots, \phi_{i_{\mathfrak{v}}}$. Assume that the last row containing a constant multiple of F is ϕ_{i_s} and label the initial coefficients of the F as c_1, \dots, c_s with at least $c_s \neq 0$. Let $\alpha = \sum_{j=1}^{\mathfrak{v}} e_{i_j}$. Then we have that the coefficient of F in $\phi_{i_t}^{(n)}$ is determined as follows:

(1) If $\deg F = 1$, then we have

$$\begin{cases} 0 & \text{if } n < s - t, \\ \binom{n}{s-t} \lambda^{n-(s-t)} & \text{otherwise.} \end{cases}$$

(2) If $\deg F > 1$, then

$$\sum_{\ell=0}^{s-t} \left(\sum_{j=0}^{n-1-\ell} \binom{n-1-j}{\ell} \lambda^{j(\alpha-1)+n-1-\ell} \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_{ij}} \right) c_{t+\ell}.$$

Proof. We will prove both statements by induction.

(1) For the base case of $n = 1$, we have from the formula

$$\begin{cases} 0 & \text{if } t < s-1, \\ 1 & \text{if } t = s-1, \\ \lambda & \text{if } t = s, \end{cases}$$

which corresponds to the linear terms of a Jordan block. We will assume now that the formula holds for the n -th iterate and consider the $(n+1)$ -st iterate.

Case 1. The Jordan block is size 1 or $t = s$.

In this case, the contribution to F in $\text{supp}(\phi_{i_s})$ through iteration is given by

$$\lambda x_s.$$

Hence, the coefficient of F in $\text{supp}(\phi_{i_s}^{(n+1)})$ is given by

$$\lambda(\lambda^n x_s) = \lambda^{n+1} x_s,$$

confirming the formula.

Case 2. The Jordan block is non-trivial and $t \neq s$.

In this case, the contribution to F in $\text{supp}(\phi_{i_t})$ through iteration is given by

$$\lambda x_t + x_{t+1},$$

and hence the coefficient of F in $\text{supp}(\phi_{i_t}^{(n+1)})$ is given by

$$\begin{aligned} & \lambda \left(\binom{n}{s-t} \lambda^{n-(s-t)} + \binom{n}{s-(t+1)} \lambda^{n-(s-(t+1))} \right) \\ &= \lambda^{n-(s-t)+1} \left(\binom{n}{s-t} + \binom{n}{s-t-1} \right) \\ &= \lambda^{n-(s-t)+1} \left(\frac{n(n-1) \cdots (s-t+1)}{(n-(s-t))!} + \frac{n(n-1) \cdots (s-t+1)(s-t)}{(n-(s-t)+1)!} \right) \\ &= \lambda^{n-(s-t)+1} \left(\frac{n(n-1) \cdots (s-t+1)(n+1-(s-t))}{(n-(s-t))!(n+1-(s-t))} + \frac{n(n-1) \cdots (s-t+1)(s-t)}{(n-(s-t)+1)!} \right) \\ &= \lambda^{n-(s-t)+1} \left(\frac{(n+1)n(n-1) \cdots (s-t+1)}{(n-(s-t)+1)!} \right) \\ &= \lambda^{n+1-(s-t)} \binom{n+1}{s-t}, \end{aligned}$$

confirming the formula.

(2) For the base case of $n = 1$, the sum over j has a term only when $\ell = 0$ and, in that case, we have $\ell = j = 0$. So the coefficient of $c_{t+\ell}$ in ϕ_{i_t} is

$$\begin{cases} 0 & \text{if } \ell \neq 0, \\ 1 = \lambda^0 \prod_{x_i|F, i \notin \{i_1, \dots, i_v\}} \lambda_i^0 & \text{if } \ell = 0. \end{cases}$$

We will assume now that the formula holds for the n -th iterate and consider the $(n+1)$ -st iterate.

Case 1. The Jordan block is size 1 or $t = s$ (in other words, $\ell = 0$).

In this case, the contribution to F in $\text{supp}(\phi_{i_s})$ through iteration is given by

$$\lambda x_s + c_s F,$$

and, hence, the coefficient in F in $\text{supp}(\phi_{i_s}^{(n+1)})$ is given by

$$(2.1) \quad \lambda \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j} \right) + (\lambda^n)^\alpha \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} (\lambda_i^n)^{e_i}.$$

Notice that the first term of (2.1) and the desired term of

$$(2.2) \quad \sum_{j=0}^n \lambda^{j(\alpha-1)+n} \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j}$$

differ by exactly the $j = n$ term of (2.2), which is exactly the second term of (2.1), confirming the formula.

Case 2. The Jordan block is non-trivial and $t \neq s$.

In this case, the contribution to F in $\text{supp}(\phi_{i_t})$ through iteration is given by

$$(2.3) \quad \lambda x_t + x_{t+1} + c_t F.$$

Case 2.1. $\ell = 0$.

Since there is no contribution to c_t from x_{t+1} in (2.3), the contribution of c_t is given by

$$(2.4) \quad \lambda \left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j} \right) + (\lambda^n)^\alpha \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} (\lambda_i^n)^{e_i}.$$

Notice that the first term of (2.4) and the desired term of

$$(2.5) \quad \sum_{j=0}^n \lambda^{j(\alpha-1)+n} \prod_{\substack{x_i|F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j}$$

differ by exactly the $j = n$ term of (2.5), which is exactly the second term of (2.4), confirming the formula.

Case 2.2. $\ell \neq 0$.

Since there is no contribution to $c_{t+\ell}$ from $c_t F$ in (2.3) for $\ell \neq 0$, the contribution of each $c_{t+\ell}$ is given by

$$(2.6) \quad \lambda \left(\sum_{j=0}^{n-1-\ell} \binom{n-1-j}{\ell} \lambda^{j(\alpha-1)+n-1-\ell} \prod_{\substack{x_i | F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j} \right)$$

$$(2.7) \quad + \sum_{j=0}^{n-1-\ell+1} \binom{n-1-j}{\ell-1} \lambda^{j(\alpha-1)+n-1-\ell+1} \prod_{\substack{x_i | F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j}.$$

For $j = n - \ell$ there is no contribution from (2.6), so we have to check that

$$\binom{n-j}{\ell} = \binom{n-1-j}{\ell-1}.$$

Computing, we get

$$\binom{\ell}{\ell} = \binom{\ell-1}{\ell-1}$$

with equality since ℓ is at least 1.

We have left to check that for $j = 0, \dots, n - \ell - 1$ we have

$$\binom{n-j}{\ell} = \binom{n-1-j}{\ell} + \binom{n-1-j}{\ell-1}.$$

Computing the right-hand side, we have

$$\begin{aligned} & \frac{(n-1-j) \cdots (\ell+1)}{(n-j-\ell-1)!} + \frac{(n-1-j) \cdots (\ell)}{(n-j-\ell)!} \\ &= \frac{(n-1-j) \cdots (\ell+1)(n-j-\ell)}{(n-j-\ell)!} + \frac{(n-1-j) \cdots (\ell+1)(\ell)}{(n-j-\ell)!} \\ &= \frac{(n-1-j) \cdots (\ell+1)(n-j-\ell+\ell)}{(n-j-\ell)!} \\ &= \frac{(n-j)(n-j-1) \cdots (\ell+1)}{(n-j-\ell)!} \\ &= \binom{n-j}{\ell}, \end{aligned}$$

confirming the formula. □

Remark. If $F \in \text{supp}(\phi_i)$ with $\lambda_i = 0$ then we know that F does not effect $LT(I_1)$ since x_i either divides $LT(f)$ or is relatively prime to $LT(f)$ for all $f \in I_1$. In the former, case we take the normal form of f with respect to the known leading terms. In the latter case, we see that every term in the local analogue of the S -polynomials is divisible by the known leading terms and hence is already in the leading term ideal.

If $F \in \text{supp}(\phi_j)$ with $x_i \mid F$ and $\lambda_i = 0$ then we know that F does not effect $LT(I_1)$ since $x_i \in LT(I_n)$ for all n .

So we will exclude from consideration the Jordan block(s) with eigenvalue 0 and monomials divisible by x_i with $\lambda_i = 0$.

Lemma 2.16. *Let $F = \prod x_i^{e_i}$ with $\deg F > 1$ and assume that $\text{supp}(\phi_{i_t})$ with $1 \leq i_t \leq b$ contains a least monomial that is a constant multiple of F . Assume that ϕ_{i_t} is in a Jordan block of $d\phi_P$ of size $\mathfrak{v} \geq 1$ with eigenvalue $\lambda \neq 0$. Label the rows $1, \dots, \mathfrak{v}$ corresponding to $\phi_{i_1}, \dots, \phi_{i_{\mathfrak{v}}}$. Let $\alpha = \sum_{j=1}^{\mathfrak{v}} e_{i_j}$. The following are conditions for all of the coefficients of F in the Jordan block of $d\phi_P$ containing F to be divisible by $p = \text{char}(K)$.*

- (1) *If $\lambda = 1$ and $\lambda_i = 1$ for all i such that $x_i \mid F$ and $i \notin \{i_1, \dots, i_{\mathfrak{v}}\}$, then it is necessary that $p \mid n$ and sufficient that $n = Mp^e$ for e large enough and $(M, p) = 1$.*
- (2) *Assume $\lambda \neq 1$ and $\lambda_i = 1$ for all i such that $x_i \mid F$ and $i \notin \{i_1, \dots, i_{\mathfrak{v}}\}$.*
 - (a) *If $\alpha = 0$, then it is necessary that λ is an r -th root of unity for some $r \mid n$ and sufficient that $n = Mrp^e$ for e large enough.*
 - (b) *If $\alpha > 0$, then it is necessary that $\lambda^{\alpha-1}$ is an r -th root of unity for some $r \mid n$.*
- (3) *If $\lambda = 1$ and $\lambda_i \neq 1$ for at least one i such that $x_i \mid F$, then it is necessary that*

$$\prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$

is an r -th root of unity for some $r \mid n$.

- (4) *If $\lambda \neq 1$ and $\lambda_i \neq 1$ for at least one i such that $x_i \mid F$ and $i \notin \{i_1, \dots, i_{\mathfrak{v}}\}$, then it is necessary that*

$$\lambda^{\alpha-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_{\mathfrak{v}}\}}} \lambda_i^{e_i}$$

is an r -th root of unity with $r \mid n$.

Proof. We will use the description of the coefficients of F under iteration from Lemma 2.15. Assume that the last row containing F is ϕ_{i_s} with $1 \leq s \leq \mathfrak{v}$ and label the initial coefficients of the F as c_1, \dots, c_s with at least $c_s \neq 0$.

If $\deg F = 1$, then $F = x_{i_s}$ and $c_s = \lambda$ and the coefficient in $\phi_{i_s}^{(n)}$ is

$$\lambda^n c_s.$$

Since $\lambda \neq 0$ in K this coefficient is never divisible by p . So we restrict to the case $\deg F > 1$.

- (1) We want the coefficients to be 0 in K . The coefficient in ϕ_{i_s} is given by

$$\left(\sum_{j=0}^{n-1} 1 \right) c_s = nc_s$$

with $c_s \neq 0$ in K . Hence, we must have $p \mid n$. To see the sufficient condition; assume that $p \mid n$, then at least $F \notin \text{supp}(\phi_{i_s}^{(n)})$. Note that if $p \neq 2$, then we also $F \notin \text{supp}(\phi_{i_{s-1}}^{(n)})$ since the coefficient

$$\left(\sum_{j=1}^{n-1} j \right) c_{s-1} = \frac{n(n-1)}{2} c_{s-1}$$

is divisible by p for $p \mid n$. Now replace ϕ by ϕ^p and n by n/p and consider F in the Jordan block of $d\phi_P$. Now the last row containing F is ϕ_{i_s} . By the above argument, this coefficient in ϕ^p will be 0 in K . Hence, with each power of p , F does not appear in the last previously appearing row after iteration. Since the Jordan block is of finite size, taking ϕ^{p^e} for e large enough causes the monomial F to not appear in the Jordan block of $d\phi_P^n$. So it is necessary that $p \mid n$ and sufficient that $n = Mp^e$ for e large enough and $(M, p) = 1$.

- (2) (a) We want the coefficients to be 0 in K . The coefficient in ϕ_{i_s} is given by

$$\left(\sum_{j=0}^{n-1} \lambda^j \right) c_s$$

with $c_s \neq 0$ in K . Hence, we must have

$$\lambda^n \equiv 1 \pmod{p}.$$

To see the sufficient condition, assume that $\lambda^r \equiv 1 \pmod{p}$ for some $r \mid n$ and replace ϕ by ϕ^r and n by n/r . Now we are in the situation of (a), which we have already demonstrated. So it is necessary that λ is an r -th root of unity for some $r \mid n$ and sufficient that $n = Mrp^e$ for e large enough and $(M, p) = 1$.

- (b) We want the coefficients to be 0 in K . The coefficient in ϕ_{i_s} is given by

$$\left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \right) c_s$$

with $c_s \neq 0$ in K . Hence, $\lambda^{\alpha-1}$ must be an r -th root of unity for some $r \mid n$.

- (3) We want the coefficients to be 0 in K . The coefficient in ϕ_{i_s} is given by

$$\left(\sum_{j=0}^{n-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j} \right) c_s$$

with $c_s \neq 0$ in K . Hence,

$$\prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i}$$

must be an r -th root of unity modulo p for some $r \mid n$.

- (4) We want the coefficients to be 0 in K . The coefficient in ϕ_{i_s} is given by

$$\left(\sum_{j=0}^{n-1} \lambda^{j(\alpha-1)+n-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i j} \right) c_s$$

Hence,

$$\lambda^{\alpha-1} \prod_{\substack{x_i \mid F \\ i \notin \{i_1, \dots, i_v\}}} \lambda_i^{e_i}$$

must be an r -th root of unity modulo p for some $r \mid n$.

□

We have now established necessary conditions for a least monomial in

$$\text{supp}(\phi_1(\mathbf{x}) - x_1, \dots, \phi_b(\mathbf{x}) - x_b)$$

to not appear in

$$\text{supp}(\phi_1^{(n)}(\mathbf{x}) - x_1, \dots, \phi_b^{(n)}(\mathbf{x}) - x_b).$$

However, this vanishing is not sufficient for $a_P(n) \neq a_P(1)$. Fortunately, the necessary conditions on n from Lemma 2.16 will be enough to show that $\Phi_n^*(\phi)$ is an effective zero-cycle for all $n \geq 1$.

The next proposition gathers our knowledge of $a_P(n)$.

Proposition 2.17. *Let $X \subset \mathbb{P}_K^N$ be a non-singular, irreducible, projective variety of dimension b defined over K . Let $\phi : X \rightarrow X$ be a morphism defined over K and $P \in X(K)$ be a fixed point of ϕ . Denote $d\phi_P$ as the map induced by ϕ on the cotangent space of X at P . Let $\lambda_1, \dots, \lambda_l$ be the distinct eigenvalues of $d\phi_P$ with primitive multiplicative orders r_1, \dots, r_l (set $r_i = \infty$ if λ_i is not a root of unity). Then for all $n \geq 1$ such that ϕ^n is non-degenerate,*

- (1) $a_P(n) \geq a_P(1)$.
- (2) $a_P(n) = 1 \Leftrightarrow \lambda_i^n \neq 1$ for $1 \leq i \leq l$.
- (3) If $a_P(n) > a_P(1)$, then at least one of the following is true.
 - (a) $r_i \mid n$ for at least one i for $1 \leq i \leq l$ with $r_i \neq 1$.
 - (b) $p = \text{char}(K) \neq 0$ and $p \mid n$.

Proof.

- (1) Lemma 2.10.
- (2) It is clear that $a_P(n) = 1$ if and only if I_n generates the maximal ideal of \widehat{R}_P . This is true if and only if

$$\{x_1 - y_1, \dots, x_b - y_b, \phi_1^{(n)}(\mathbf{x}) - y_1, \dots, \phi_b^{(n)}(\mathbf{x}) - y_b\}$$

is a regular local system of parameters. Zariski and Samuel [21, Corollary 2 page 137] state that this occurs if and only if the power series

$$\{x_1 - y_1, \dots, x_b - y_b, \phi_1^{(n)}(\mathbf{x}) - y_1, \dots, \phi_b^{(n)}(\mathbf{x}) - y_b\}$$

contain independent linear terms. This is true if and only if $\lambda_i^n \neq 1$ for $1 \leq i \leq l$.

- (3) We know from Corollary 2.9 that $a_P(n) > a_P(1)$ if and only if certain monomials F has zero coefficients after iteration. Any such monomial must be a least monomial by Lemma 2.11 and Lemma 2.13. Lemma 2.16 gives necessary conditions on n for which any least monomial has zero coefficients after iteration. Note that cases (2b), (3), and (4) of Lemma 2.16 are cases where $a_P(n) = a_P(1)$ since $\lambda_i \neq 1$ for some $x_i \mid F$. Hence, the removal of this monomial has no effect on the leading term ideal. So we are concerned only with the conditions (1) and (2a) of Lemma 2.16 for which we also know sufficient conditions.

□

2.3. Proof of effectivity. We will consider several different maps over the course of the proof, so to avoid confusion we will include the map in the notation as $a_P(\phi, n)$ and $a_P^*(\phi, n)$.

Lemma 2.18. *Let p be a prime in \mathbb{Z} and let $n = Mp^e$ in \mathbb{Z}^+ with $e \geq 1$ and $p \nmid M$.*

- (1) If $e = 1$, then

$$a_P^*(\phi, n) = a_P^*(\phi^p, M) - a_P^*(\phi, M).$$

- (2) If $e \geq 2$, then

$$a_P^*(\phi, n) = a_P^*(\phi^{p^{e-1}}, Mp).$$

- (3) Let $n = qM$ where $\gcd(q, M) = 1$. Then

$$a_P^*(\phi, n) = \sum_{d \mid q} \mu\left(\frac{q}{d}\right) a_P^*(\phi^d, M).$$

Proof. Computing, we get

$$\begin{aligned}
a_P^*(\phi, n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\
&= \sum_{pd|n} \mu\left(\frac{n}{pd}\right) a_P(\phi, pd) + \sum_{d|M} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\
&= \sum_{d|Mp^{e-1}} \mu\left(\frac{Mp^{e-1}}{d}\right) a_P(\phi^p, d) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d) \\
&= a_P^*(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d).
\end{aligned}$$

So we have

$$(2.8) \quad a_P^*(\phi, n) = a_P^*(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d).$$

(1) Considering (2.8) with $e = 1$, we have

$$\begin{aligned}
a_P^*(\phi, n) &= a_P^*(\phi^p, M) + \sum_{d|M} \mu\left(\frac{Mp}{d}\right) a_P(\phi, d) \\
&= a_P^*(\phi^p, M) + \sum_{d|M} \mu(p) \mu\left(\frac{M}{d}\right) a_P(\phi, d) \\
&= a_P^*(\phi^p, M) - a_P^*(\phi, M),
\end{aligned}$$

where the middle equality comes from the fact that μ is multiplicative and $(p, M) = 1$.

(2) Considering (2.8) with $e > 1$, we have

$$\begin{aligned}
a_P^*(\phi, n) &= a_P^*(\phi^p, Mp^{e-1}) + \sum_{d|M} \mu\left(\frac{Mp^e}{d}\right) a_P(\phi, d) \\
&= a_P^*(\phi^p, Mp^{e-1}) + 0,
\end{aligned}$$

where the second equality comes from the fact that $\frac{Mp^e}{d}$ is not square free for all $d | M$. Replacing ϕ by ϕ^p and n by n/p , we may repeat the argument to conclude that

$$a_P^*(\phi, n) = a_P^*(\phi^{p^{e-1}}, Mp).$$

(3) Using the fact that the Möbius function is multiplicative for relatively prime numbers, we get

$$\begin{aligned}
a_P^*(\phi, n) &= \sum_{d|qM} \mu\left(\frac{qM}{d}\right) a_P(\phi, d) \\
&= \sum_{d_1|q} \sum_{d_2|M} \mu\left(\frac{qM}{d_1 d_2}\right) a_P(\phi, d_1 d_2) \\
&= \sum_{d_1|q} \sum_{d_2|M} \mu\left(\frac{q}{d_1}\right) \mu\left(\frac{M}{d_2}\right) a_P(\phi, d_1 d_2) \\
&= \sum_{d_1|q} \mu\left(\frac{q}{d_1}\right) \sum_{d_2|M} \mu\left(\frac{M}{d_2}\right) a_P(\phi^{d_1}, d_2) \\
&= \sum_{d_1|q} \mu\left(\frac{q}{d_1}\right) a_P^*(\phi^{d_1}, M).
\end{aligned}$$

□

In the next lemma, we provide a formula for $a_P^*(n)$ when $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})$ for some subset $\{r_{i_1}, \dots, r_{i_k}\}$ of $\{r_1, \dots, r_l\}$. We clearly need that each r_{i_t} is finite, in other words, that λ_{i_t} has finite order, and we will also assume that each $r_{i_t} \neq 1$.

Lemma 2.19. *Let P be a fixed point of ϕ . Let r_i be the primitive order of λ_i in K^* for $1 \leq i \leq l$ (set $r_i = \infty$ if λ_i is not a root of unity). If $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})$ for some subset of non-trivial finite orders $\{r_{i_1}, \dots, r_{i_k}\} \subseteq \{r_1, \dots, r_l\}$ with $n \neq 0$ in K and square free with no other r_i dividing n , then we have*

$$\begin{aligned}
a_P^*(\phi, n) &= \\
&\sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(1) \\
&+ \sum_{t=1}^k \sum_{d|\frac{n}{r_{i_t}}} \mu\left(\frac{n}{d r_{i_t}}\right) c_{i_t} \\
&+ \sum_{t1=1}^k \sum_{\substack{t2=1 \\ i_{t2} \neq i_{t1}}}^k \sum_{d|\frac{n}{\text{lcm}(r_{i_{t1}}, r_{i_{t2}})}} \mu\left(\frac{n}{d \text{lcm}(r_{i_{t1}}, r_{i_{t2}})}\right) c_{i_{t1}, i_{t2}} \\
&\vdots \\
&+ \sum_{t=1}^k \sum_{d|\frac{n}{\text{lcm}(r_{i_1}, \dots, \widehat{r_{i_t}}, \dots, r_{i_k})}} \mu\left(\frac{n}{d \text{lcm}(r_{i_1}, \dots, \widehat{r_{i_t}}, \dots, r_{i_k})}\right) c_{i_1, \dots, \widehat{c_{i_t}}, \dots, i_k} \\
&+ \sum_{d|1} c_{i_1, \dots, i_k}
\end{aligned}$$

for some non-negative constants c_α .

Proof. Recall that

$$LT(I_{\Gamma_n} + I_\Delta) = LT(I_n) \subseteq LT(I_1) = LT(I_{\Gamma_1} + I_\Delta).$$

In particular, we know that

$$\text{Span}(X^v \mid X^v \notin LT(I_1)) \subseteq \text{Span}(X^v \mid X^v \notin LT(I_n)).$$

From Proposition 2.17, we know that $a_P(r_{i_t}) > a_P(1)$ for each r_{i_t} since $r_{i_t} \neq 1$. Similarly for $i_{t1} \neq i_{t2}$, by replacing ϕ with $\phi^{r_{i_{t1}}}$, we have

$$\begin{cases} a_P(\text{lcm}(r_{i_{t1}}, r_{i_{t2}})) > a_P(r_{i_{t1}}) & \text{if } r_{i_{t2}} \nmid r_{i_{t1}} \\ a_P(\text{lcm}(r_{i_{t1}}, r_{i_{t2}})) = a_P(r_{i_{t1}}) & \text{if } r_{i_{t2}} \mid r_{i_{t1}} \end{cases}$$

since in the second case $\text{lcm}(r_{i_{t1}}, r_{i_{t2}}) = r_{i_{t1}}$. Continuing in the same manner, we have

$$\begin{cases} a_P(\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma})) > a_P(\text{lcm}(r_{i_1}, \dots, r_{i_j})) & \text{if } r_{i_\gamma} \nmid \text{lcm}(r_{i_1}, \dots, r_{i_j}) \\ a_P(\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma})) = a_P(\text{lcm}(r_{i_1}, \dots, r_{i_j})) & \text{if } r_{i_\gamma} \mid \text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma}). \end{cases}$$

Again in the second case, we have $\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma}) = \text{lcm}(r_{i_1}, \dots, r_{i_j})$, so we have left to consider the first case. In particular, for any β defined as the least common multiple of any j of $\{r_{i_1}, \dots, r_{i_j}, r_{i_\gamma}\}$, we have $\{X^v \mid X^v \notin LT(I_{\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma})})\}$ containing at least one element not in $\{X^v \mid X^v \notin LT(I_\beta)\}$. To see this, consider the ordering

$$x_{i_1} < x_{i_2} < \dots < x_{i_j} < x_\gamma < x_{i_t} < \dots < x_{i_{b-j-1}}.$$

Then for each β , one of the linear terms $x_{i_1}, \dots, x_{i_j}, x_{i_\gamma}$ is contained in $LT(I_\beta)$ since it is a leading term of the associated $\phi_i^{(\beta)}(x_1, \dots, x_b) - x_i$. Also, none of the linear terms $x_{i_1}, \dots, x_{i_j}, x_{i_\gamma}$ are contained in $LT(I_{\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma})})$. Hence, the monomial

$$x_{i_1} x_{i_2} \dots x_{i_j} x_\gamma$$

is in $\{X^v \mid X^v \notin LT(I_{\text{lcm}(r_{i_1}, \dots, r_{i_j}, r_{i_\gamma})})\}$ but not in any of the $\{X^v \mid X^v \notin LT(I_\beta)\}$. This argument ensures the non-negativity of the constants c_α defined below.

We have $a_P(1) \geq 1$ since P is a fixed point and since

$$\{X^v \mid X^v \notin LT(I_1)\} \subseteq \{X^v \mid X^v \notin LT(I_\kappa)\}$$

for all $\kappa \geq 1$, we have a contribution of $a_P(1)$ to $a_P(d)$ for all $d \mid n$.

Let $c_{i_t} = a_P(r_{i_t}) - a_P(1) > 0$ for $1 \leq t \leq k$ since $r_{i_t} \neq 1$ by assumption for $1 \leq t \leq k$. Since

$$\{X^v \mid X^v \notin LT(I_{r_{i_t}})\} \subseteq \{X^v \mid X^v \notin LT(I_\kappa)\}$$

for all κ with $r_{i_t} \mid \kappa$, we have a contribution of c_{i_t} to $a_P(d)$ for all $d \mid \frac{n}{r_{i_t}}$.

Let

$$c_{i_{t1}, i_{t2}} = a_P(\text{lcm}(r_{i_{t1}}, r_{i_{t2}})) - \#\{X^v \mid X^v \notin LT(I_{r_{i_{t1}}})\} \cup \{X^v \mid X^v \notin LT(I_{r_{i_{t2}}})\}.$$

If $r_{i_{t2}} \mid r_{i_{t1}}$, then $c_{i_{t1}, i_{t2}} = 0$ since $\text{lcm}(r_{i_{t1}}, r_{i_{t2}}) = r_{i_{t1}}$. Otherwise, by the argument at the beginning of the proof, there is at least one monomial not in $LT(I_{r_{t1}, r_{t2}})$ that is not in the complement of $LT(I_{r_{t1}})$ or $LT(I_{r_{t2}})$. Hence $c_{i_{t1}, i_{t2}} \geq 0$. Since

$$\{X^v \mid X^v \notin LT(I_{\text{lcm}(r_{i_{t1}}, r_{i_{t2}})})\} \subseteq \{X^v \mid X^v \notin LT(I_\kappa)\}$$

for all κ with $\text{lcm}(r_{i_{t1}}, r_{i_{t2}}) \mid \kappa$, we have a contribution of $c_{i_{t1}, i_{t2}}$ to $a_P(d)$ for all $d \mid \frac{n}{\text{lcm}(r_{i_{t1}}, r_{i_{t2}})}$.

Similarly, for $2 \leq j \leq k$, let β be the least common multiple of j elements of $\{r_{i_{t1}}, \dots, r_{i_{tj}}, r_{i_\gamma}\}$ and let

$$c_{i_{t1}, \dots, i_{tj}, i_\gamma} = a_P(\text{lcm}(r_{i_{t1}}, \dots, r_{i_{tj}}, r_{i_\gamma})) - \# \left(\bigcup_{\beta} \{X^v \mid X^v \notin LT(I_\beta)\} \right).$$

If $r_{i_\gamma} \mid \text{lcm}(r_{i_{t1}}, \dots, r_{i_{tj}})$, then $c_{i_{t1}, \dots, i_{tj}, i_\gamma} = 0$ since $\text{lcm}(r_{i_{t1}}, \dots, r_{i_{tj}}, r_{i_\gamma}) = \text{lcm}(r_{i_{t1}}, \dots, r_{i_{tj}})$. Otherwise, by the argument at the beginning of the proof, there is at least one monomial not in $LT(I_{r_{i_{t1}}, \dots, r_{i_{tj}}, r_{i_\gamma}})$ that is not in the complement of $LT(I_\beta)$ for each β . Hence $c_{i_{t1}, \dots, i_{tj}, i_\gamma} \geq 0$. Since

$$\{X^v \mid X^v \notin LT(I_{\text{lcm}(r_{i_{t1}}, \dots, r_{i_{tj}}, r_{i_\gamma})})\} \subseteq \{X^v \mid X^v \notin LT(I_\kappa)\}$$

for all κ with $\text{lcm}(r_{i_{t_1}}, \dots, r_{i_{t_j}}, r_{i_\gamma}) \mid \kappa$, we have a contribution of $c_{i_{t_1}, \dots, i_{t_j}, i_\gamma}$ to $a_P(d)$ for all $d \mid \frac{n}{\text{lcm}(r_{i_{t_1}}, \dots, r_{i_{t_j}}, r_{i_\gamma})}$.

Notice that by construction, none of the monomials in $\{X^v \mid X^v \notin LT(I_n)\}$ are counted in multiple constants c_α , and all of them have been counted. Hence, the formula holds. \square

Remark. Notice that Lemma 2.19 implies that $a_P^*(n) \geq 0$ for all $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})$ since each line is either 0 or c_α by properties of the Möbius function and the constants c_α are all non-negative. We may assume n is square free by Lemma 2.18(2).

We are now ready to prove the main theorem.

Theorem 2.20. *Let $X \subset \mathbb{P}_K^N$ be a non-singular, irreducible, projective variety of dimension b defined over K and let $\phi : X \rightarrow X$ be a morphism defined over K . Let P be a point in $X(K)$. Define integers*

$p =$ the characteristic of K .

$m =$ the primitive period of P for ϕ (set $m = \infty$ if $P \notin \text{Per}(\phi)$).

If m is finite, let $d\phi_P^m$ be the map induced by ϕ^m on the cotangent space of X at P , and let $\lambda_1, \dots, \lambda_l$ be the distinct eigenvalues of $d\phi_P^m$. Define

$r_i =$ the multiplicative period of λ_i in K^ (set $r_i = \infty$ if λ_i is not a root of unity).*

Then

- (1) *For all $n \geq 1$ such that ϕ^n is non-degenerate, $a_P^*(n) \geq 0$.*
- (2) *Let $n \geq 1$. If ϕ^n is non-degenerate and $a_P^*(n) \geq 1$, then n has one of the following forms:*
 - (a) $n = m$.
 - (b) $n = m \text{lcm}(r_{i_1}, \dots, r_{i_k})$ for $1 \leq k \leq l$.
 - (c) $n = m \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e$ for $1 \leq k \leq l$ and some $e \geq 1$.

Proof. Fix a point $P \in X$ and let $n \geq 1$ be an integer such that ϕ^n is non-degenerate. By definition, we have

$$a_P^*(\phi, n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(\phi, d).$$

Suppose that $\phi^n(P) \neq P$. Then $\phi^d(P) \neq P$ for all $d \mid n$, so $a_P(\phi, d) = 0$ for all $d \mid n$ since the graph Γ_d of ϕ^d and the diagonal Δ will not intersect at (P, P) . Hence, $a_P^*(\phi, n) = 0$, proving the theorem in this situation. We now assume that $\phi^n(P) = P$.

It follows that P is a periodic point for ϕ , so m is finite with $m \mid n$ and $a_P(\phi, d) \geq 1$ if and only if $m \mid d$. Computing $a_P^*(\phi, n)$ in terms of ϕ^m , we see that

$$\begin{aligned} a_P^*(\phi, n) &= \sum_{d \mid n \text{ with } m \mid d} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\ &= \sum_{d \mid (n/m)} \mu\left(\frac{n}{md}\right) a_P(\phi, md) \\ &= \sum_{d \mid (n/m)} \mu\left(\frac{n/m}{d}\right) a_P(\phi^m, d) \\ &= a_P^*(\phi^m, n/m). \end{aligned}$$

Therefore, we can replace ϕ by ϕ^m and n by n/m and assume that $m = 1$.

We will consider a number of cases, but first we recall from Proposition 2.17 that $a_P(\phi, 1) = 1$ if and only if $r_i \neq 1$ for all $1 \leq i \leq l$.

Case 1. $n = 1$, in other words $n = m$.

In this case, we have

$$a_P^*(\phi, n) = a_P(\phi, 1).$$

Since P is assumed to be fixed by ϕ ,

$$a_P^*(\phi, n) = a_P(\phi, 1) \geq \begin{cases} 1 & \text{always} \\ 2 & \text{if } r_i = 1 \text{ for some } i. \end{cases}$$

Case 2. $n > 1$ and $a_P(\phi, n) = a_P(\phi, 1)$.

Let $d \mid n$; then Proposition 2.17 states that

$$a_P(\phi, 1) = a_P(\phi, n) = a_P(\phi^d, n/d) \geq a_P(\phi^d, 1) = a_P(\phi, d) \geq a_P(\phi, 1).$$

Hence, $a_P(\phi, d) = a_P(\phi, 1)$ for all $d \mid n$. So

$$a_P^*(\phi, n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) a_P(\phi, 1) = 0$$

by properties of the Möbius function, since $n > 1$ by assumption.

Case 3. $a_P(\phi, n) > a_P(\phi, 1)$ and $n \neq 0$ in K .

By the assumptions in this case, we know that at least one $r_i \mid n$. Let $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})M$ where M is not divisible by any r_i . Then we have

$$a_P^*(\phi, n) = \sum_{d \mid M} a_P^*(\phi^d, \text{lcm}(r_{i_1}, \dots, r_{i_k}))$$

by Lemma 2.18(3). However, since $r_i \nmid M$ for all $1 \leq i \leq l$, for $d \mid M$, we also have $r_i \nmid d$ for all $1 \leq i \leq l$. Additionally, $n \neq 0$ implies $p \nmid n$, so we cannot be in any condition of Proposition 2.17(3). Consequently,

$$\begin{aligned} a_P^*(\phi, n) &= \sum_{d \mid M} \mu\left(\frac{M}{d}\right) a_P^*(\phi^d, \text{lcm}(r_{i_1}, \dots, r_{i_k})) \\ &= \sum_{d \mid M} \mu\left(\frac{M}{d}\right) a_P^*(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})) \\ &= 0 \end{aligned}$$

since $a_P^*(\phi^d, \text{lcm}(r_{i_1}, \dots, r_{i_k}))$ is constant over $d \mid M$. So we can assume that $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})$ and $r_i \nmid n$ for $i \notin \{i_1, \dots, i_k\}$. If n is not square free, then by applying Lemma 2.18 to any prime factor $q_j^{e_j}$ with $e_j > 1$, we get

$$a_P^*(\phi, n) = a_P^*\left(\phi^{q_1^{e_1-1} \dots q_j^{e_j-1}}, \frac{n}{q_1^{e_1-1} \dots q_j^{e_j-1}}\right).$$

So we may replace n by $\frac{n}{q_1^{e_1-1} \dots q_j^{e_j-1}}$ and ϕ by $\phi^{q_1^{e_1-1} \dots q_j^{e_j-1}}$ and assume that n is square free. We are now in the case of Lemma 2.19 and have

$$\begin{aligned}
a_P^*(\phi, n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(1) \\
&+ \sum_{t=1}^k \sum_{d|\frac{n}{r_{i_t}}} \mu\left(\frac{n}{dr_{i_t}}\right) c_{i_t} \\
&+ \sum_{t=1}^k \sum_{\substack{t_2=1 \\ i_{t_2} \neq i_{t_1}}}^k \sum_{d|\frac{n}{\text{lcm}(r_{i_{t_1}}, r_{i_{t_2}})}} \mu\left(\frac{n}{d \text{lcm}(r_{i_{t_1}}, r_{i_{t_2}})}\right) c_{i_{t_1}, i_{t_2}} \\
&\vdots \\
&+ \sum_{t=1}^k \sum_{d|\frac{n}{\text{lcm}(r_{i_1}, \dots, r_{i_t}, \dots, r_{i_k})}} \mu\left(\frac{n}{d \text{lcm}(r_{i_1}, \dots, r_{i_t}, \dots, r_{i_k})}\right) c_{i_1, \dots, \widehat{c_{i_t}}, \dots, i_k} \\
&+ \sum_{d|1} c_{i_1, \dots, i_k}
\end{aligned}$$

for some non-negative constants c_α . Since every inner sum is either 0 or c_α by properties of the Möbius function, we have that $a_P^*(\phi, n) \geq 0$ because every c_α is non-negative. By assumption, at least one r_i divides n , so we know that c_n will be positive since it will have at least one additional monomial. Additionally, the sum associated to c_n will be c_n since it is summing over the divisors of 1. So we have shown that

$$\begin{cases} a_P^*(\phi, n) \geq 1 & \text{if } M=1 \\ a_P^*(\phi, n) = 0 & \text{otherwise.} \end{cases}$$

Case 4. $a_P(\phi, n) > a_P(\phi, 1)$ and $n = 0$ in K .

We can write $n = \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e M$ with $(r_{i_t}, M) = 1 = (p, M)$ by Proposition 2.17. If $M > 1$, then

$$a_P^*(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e) = a_P^*(\phi, d \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e)$$

for all $d \mid M$ since M is not in one of the forms of Proposition 2.17(3). So

$$\begin{aligned}
a_P^*(\phi, n) &= \sum_{d|M} \mu\left(\frac{n}{d}\right) a_P(\phi^d, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e) \\
&= \sum_{d|M} \mu\left(\frac{n}{d}\right) a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e) \\
&= 0,
\end{aligned}$$

where the first equality is from Lemma 2.18(3). So assume $M = 1$. Computing, we have

$$\begin{aligned}
a_P^*(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p^e) &= a_P^*(\phi^{p^{e-1}}, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p) \\
&= a_P^*(\phi^{p^{e-1}}, \text{lcm}(r_{i_1}, \dots, r_{i_k}) p) - a_P^*(\phi^{p^{e-1}}, \text{lcm}(r_{i_1}, \dots, r_{i_k})) \\
&= a_P^*(\phi^{p^e}, \text{lcm}(r_{i_1}, \dots, r_{i_k})) - a_P^*(\phi^{p^{e-1}}, \text{lcm}(r_{i_1}, \dots, r_{i_k})).
\end{aligned}$$

Considering the maps ϕ^{p^e} and $\phi^{p^{e-1}}$, we have $\text{lcm}(r_{i_1}, \dots, r_{i_k}) \neq 0$ in K . As in Case 3, we may assume that $\text{lcm}(r_{i_1}, \dots, r_{i_k})$ is square free and use Lemma 2.19 to write $a_P^*(\text{lcm}(r_{i_1}, \dots, r_{i_k}))$ in terms

of the non-negative constants c_α . Since we are working with constants c_α for different maps, we include the map in the notation as $c_\alpha(\phi^{p^e})$. The constants that contribute to $a_P^*(\text{lcm}(r_{i_1}, \dots, r_{i_k}))$ are associated to $\alpha = \text{lcm}(r_{i_1}, \dots, r_{i_k})$ since the Möbius sum is not identically 0 in that case. So if

$$a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^e) = a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^{e-1}),$$

then $c_\alpha(\phi^{p^e}) = c_\alpha(\phi^{p^{e-1}})$. If we get additional key monomials with zero coefficients after iteration, in other words,

$$a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^e) > a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^{e-1}),$$

then $c_\alpha(\phi^{p^e}) > c_\alpha(\phi^{p^{e-1}})$. Hence,

$$\begin{cases} a_P^*(\phi, n) = 0 & \text{if } a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^e) = a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^{e-1}) \\ a_P^*(\phi, n) > 0 & \text{if } a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^e) > a_P(\phi, \text{lcm}(r_{i_1}, \dots, r_{i_k})p^{e-1}). \end{cases}$$

Hence, $a_P^*(\phi, n) \geq 0$ always; and if $a_P^*(\phi, n) > 0$, then n is in one of the stated forms. \square

Remark. If $\text{char } K = 0$, then in Theorem 2.20 we, in fact, have $a_P^*(n) \geq 1$ if and only if $n = m$ or $n = m \text{lcm}(r_{i_1}, \dots, r_{i_k})$ since we know precisely the conditions for $a_P(n) > a_P(1)$.

Note that Morton and Silverman [13, Corollary 3.3] show that for $\dim X = b = 1$, if $n_1 \nmid n_2$ and $n_2 \nmid n_1$, then $\Phi_{n_1}^*(\phi)$ and $\Phi_{n_2}^*(\phi)$ have disjoint support. They use this fact to construct units in K called dynatonic units similar to the construction of cyclotomic and elliptical units. In the general case, the non-divisibility condition may not imply disjoint supports because there are more possible forms of n . In particular, $n_1 = mr_1$ and $n_2 = mr_2$ could satisfy the divisibility condition, but they do not have disjoint support.

3. FORMAL n -PERIODIC POINTS OF MULTIPLICITY ONE ARE PRIMITIVE n -PERIODIC POINTS.

In this section we use the detailed description of the multiplicities from Section 2 to show that periodic points of formal period n with $n \neq 0$ in K and multiplicity one have primitive period n , generalizing [12, Theorem 2.5].

Theorem 3.1. *If P is a primitive m -periodic point for ϕ , then $a_P^*(n) \geq 2$ for all integers $n > m$ with $\text{char } K \nmid n$ and $a_P^*(n) \neq 0$.*

Proof. Let $n > m \geq 1$ be any integer for which $a_P^*(n) \neq 0$. Since

$$a_P^*(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d)$$

and $a_P(d) \neq 0$ only for $m \mid d$, we must have m divides n . Computing $a_P^*(\phi, n)$ in terms of ϕ^m , we know that

$$a_P^*(\phi, n) = a_P^*(\phi^m, n/m).$$

Hence, we may replace ϕ by ϕ^m and n by n/m and assume that P is a fixed point. From Theorem 2.20 we know that for $a_P^*(n) \neq 0$ and $\text{char } K \nmid n$ we have that n is of the form

$$n = \text{lcm}(r_{i_1}, \dots, r_{i_k}).$$

Case 1. $n = r_i$ for some $1 \leq i \leq b$ (in other words, $k = 1$).

If λ_i is in a Jordan block of $d\phi_P$ of size > 1 then consider as i the first row of the Jordan block. Let β be the size of the Jordan block. In other words $x_i, \dots, x_{i+\beta}$ are the rows of the Jordan block. Let

$$\delta = \begin{cases} i & \beta = 1 \\ i + \beta & \beta > 1. \end{cases}$$

Case 1.1. $\lambda_j \neq 1$ and $\lambda_j^n \neq 1$ for all $j \neq i$.

We have $a_P(1) = 1$ and need to compute $a_P(n)$. We know

$$\begin{cases} x_j \in \text{supp}(\phi_j^{(n)}(\mathbf{x}) - x_j) & j \neq i, \beta = 1 \\ x_{j+1} \in \text{supp}(\phi_j^{(n)}(\mathbf{x}) - x_j) & i \leq j < i + \beta, \beta > 1 \\ x_j \in \text{supp}(\phi_j^{(n)}(\mathbf{x}) - x_j) & j \notin \{i, \dots, i + \beta\}, \beta > 1 \end{cases}$$

and, using Lemma 2.15 for the description of the coefficients of a monomial after iteration, we know that

$$(3.1) \quad x_i^2 \notin \text{supp}(\phi_\delta^{(n)}(\mathbf{x}) - x_\delta).$$

With the appropriate choice of admissible monomial ordering, we have x_j for $j \neq i$ is a leading term of one of the $\phi_k^{(n)}(\mathbf{x}) - x_k$ for $k \neq \delta$. Since all of these leading terms are relatively prime they are part of the generating set of a standard basis and we need only consider the monomial $x_i^e \in \text{supp}(\phi_\delta^{(n)}(\mathbf{x}) - x_\delta)$. From (3.1) we must have $e \geq 3$. So then we have

$$LT(I_n) \subset \{x_1, \dots, x_{i-1}, x_i^3, x_{i+1}, \dots, x_b\}.$$

By Lemma 2.19, we have added at least $\{x_i, x_i^2\}$ to the complement of the leading term ideal and so

$$a_P^*(n) \geq 2.$$

Case 1.2. $\lambda_j^n = 1$ for some $j \neq i$.

We have x_i in $LT(I_d)$ for any $d < n$ but not in $LT(I_n)$ and $x_j \notin LT(I_n)$. So we have added at least

$$\{x_i, x_i x_j\}$$

to the complement of $LT(I_n)$, and by Lemma 2.19 we have

$$a_P^*(n) \geq 2.$$

Case 2. $k > 1$.

We have that $n = \text{lcm}(r_{i_1}, \dots, r_{i_k})$.

Case 2.1. $\lambda_j \neq 1$ for $j \notin \{i_1, \dots, i_k\}$.

We have $a_P(1) = 1$. From Case 1.1 we know that $a_P(r_i) \geq 3$ for each $i \in \{i_1, \dots, i_k\}$. Hence, we add at least

$$\{x_{i_1} \cdots x_{i_k}, x_{i_1}^2 \cdots x_{i_k}\}$$

to the complement of the $LT(I_n)$. So by Lemma 2.19 we have

$$a_P^*(n) \geq 2.$$

Case 2.2. $\lambda_j = 1$ for some $j \notin \{i_1, \dots, i_k\}$.

We know $x_j \notin LT(I_1)$ and hence $x_j \notin LT(I_n)$. Additionally, $x_{i_1} \cdots x_{i_k} \in LT(I_h)$ for $h \mid n$ with $h < n$, but $x_{i_1} \cdots x_{i_k} \notin LT(I_n)$ since r_{i_t} divides n for each $1 \leq t \leq k$. Consequently, we add at least

$$\{x_{i_1} \cdots x_{i_k}, x_{i_1} \cdots x_{i_k} x_j\}$$

to the complement of $LT(I_n)$. So by Lemma 2.19 we have

$$a_P^*(n) \geq 2.$$

□

Example 3.2. Theorem 3.1 does not hold for $\text{char } K \mid n$. In other words, we may have $a_P^*(n) = 1$, but P is a periodic point of primitive period strictly less than n if $\text{char } K \mid n$. For example, consider $\text{char } K = 3$, $\dim X = 2$, and $\phi : X \rightarrow X$ defined near a fixed point P as

$$\begin{aligned} \phi_1(x_1, x_2) &= x_1 + x_1^2 + x_1 x_2 \\ \phi_2(x_1, x_2) &= 2x_2 + x_1^2. \end{aligned}$$

Then with the monomial ordering $x_2 < x_1$, the leading term ideal is generated by $\{x_1^2, x_2\}$ and, hence, $a_P(1) = 2$. Iterating, we have

$$\phi_1^{(3)}(x_1, x_2) = x_1 + x_1^3 + x_1x_2 + \text{higher order terms}$$

$$\phi_2^{(3)}(x_1, x_2) = 2x_2 + x_1^2 + \text{higher order terms}.$$

Then we have the leading term ideal is generated by $\{x_1^3, x_2\}$ and, hence, $a_P(3) = 3$. Then computing

$$a_P^*(3) = a_P(3) - a_P(1) = 1,$$

but P is a fixed point for ϕ .

4. PROPERTIES AND CONSEQUENCES

Unless otherwise stated, we assume that X is a non-singular, irreducible, projective variety of dimension b defined over K and that $\phi : X \rightarrow X$ is a morphism defined over K such that ϕ^n is non-degenerate.

4.1. Basic properties.

Proposition 4.1. *Let $m, n \geq 1$ be integers such that ϕ^{mn} is non-degenerate. Then*

- (1) *If $a_P(n) > 0$, then $a_P(mn) > 0$ for all m .*
- (2) *If P is a periodic point of primitive period n for ϕ , then $a_P^*(n) \neq 0$. In particular, points of primitive period n are points of formal period n .*
- (3) *$a_P(n) = \sum_{d|n} a_P^*(d)$.*
- (4) *If $a_P(n) > 0$, then for m the primitive period of P for ϕ we have $a_P^*(m) > 0$, for all $d < m$ we have $a_P^*(d) = 0$, and $a_P^*(\phi, n) = a_P^*(\phi^m, n/m)$.*

Proof.

- (1) The multiplicity $a_P(n) > 0$ implies that P is a periodic point of period n , and, hence, $\phi^n(P) = P$.

$$\phi^n(P) = P \Rightarrow \phi^{nm}(P) = \phi^n(\phi^n(\phi^n(\dots \phi^n(P) \dots)))(m \text{ times})$$

But $\phi^n(P) = P$, so then $\phi^{nm}(P) = P$. Hence, P is also a periodic point of period mn , so it has non-zero multiplicity in $\Phi_{mn}(\phi)$.

- (2) Since $\phi^d(P) \neq P$ for all $d < n$, we have

$$a_P(d) = 0 \quad \text{for all } d < n.$$

So we have that

$$a_P^*(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d) = a_P(n) \neq 0,$$

where the last inequality comes from the fact that P is a periodic point of period n .

- (3) The definition of $\Phi_n^*(\phi)$ is

$$\Phi_n^*(\phi) = \sum_{P \in X} a_P^*(n)(P).$$

We also have

$$a_P^*(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d).$$

We can apply Möbius inversion to get

$$a_P(n) = \sum_{d|n} a_P^*(d),$$

which gives the factorization as desired.

- (4) The multiplicity $a_P(n) > 0$ implies that $\phi^n(P) = P$ and, hence, that P is a periodic point. Consequently, P has some primitive period $m \leq n$. By (2), m satisfies $a_P^*(m) > 0$. It is the minimal such value because for any $d < m$ we have that P is not a periodic point of period d and, hence, $a_P(d) = 0$. So we have $a_P^*(d) = 0$ for $d < m$. Finally, computing $a_P^*(\phi, n)$ in terms of ϕ^m we have

$$\begin{aligned} a_P^*(\phi, n) &= \sum_{d|n \text{ with } m|d} \mu\left(\frac{n}{d}\right) a_P(\phi, d) \\ &= \sum_{d|(n/m)} \mu\left(\frac{n}{md}\right) a_P(\phi, md) \\ &= \sum_{d|n/m} \mu\left(\frac{n}{d}\right) a_P(\phi^m, d) \\ &= a_P^*(\phi^m, n/m). \end{aligned}$$

□

In the next proposition, we summarize some of the facts about $a_P^*(n)$ in terms of $\Phi_n^*(\phi)$.

Proposition 4.2. *Let $m, n \geq 1$ be integers with ϕ^{mn} non-degenerate.*

- (1) $a_P^*(\phi, mn) \geq a_P^*(\phi^m, n)$.
- (2) If $(n, m) = 1$, then $\Phi_n^*(\phi^m) = \sum_{d|m} \Phi_{nd}^*(\phi)$.
- (3) Let $m = p^e$ for some prime p and $e \geq 2$. Then $\Phi_{np^e}^*(\phi) = \Phi_{np}^*(\phi^{p^{e-1}})$.
- (4) If $n = p_1^{e_1} \cdots p_r^{e_r}$ for distinct primes p_1, \dots, p_r with $e_1, \dots, e_r \geq 2$ and $m = p_1^{e_1-1} \cdots p_r^{e_r-1}$, then $\Phi_n^*(\phi) = \Phi_{p_1 \cdots p_r}^*(\phi^m)$.

Proof.

- (1) This is clear from Lemma 2.18.
- (2) We need to see that

$$a_P^*(\phi^m, n) = \sum_{d|m} a_P^*(\phi, nd).$$

By the Möbius inversion formula, this is equivalent to

$$a_P^*(\phi, mn) = \sum_{d|m} \mu\left(\frac{m}{d}\right) a_P^*(\phi^d, n).$$

Computing the right-hand side, we have

$$\begin{aligned} \sum_{d|m} \mu\left(\frac{m}{d}\right) a_P^*(\phi^d, n) &= \sum_{d|m} \mu\left(\frac{m}{d}\right) \sum_{d'|n} \mu\left(\frac{n}{d'}\right) a_P(\phi^d, d') \\ &= \sum_{d|m} \sum_{d'|n} \mu\left(\frac{m}{d}\right) \mu\left(\frac{n}{d'}\right) a_P(\phi, dd') \\ &= \sum_{d|m} \sum_{d'|n} \mu\left(\frac{nm}{dd'}\right) a_P(\phi, dd') \\ &= \sum_{d''|nm} \mu\left(\frac{nm}{d''}\right) a_P(\phi, d'') \\ &= a_P^*(\phi, nm). \end{aligned}$$

- (3) This is Lemma 2.18(2).
- (4) This is Lemma 2.18(2) applied to each p_i .

□

Proposition 4.3. $\deg(\Phi_n^*(\phi)) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \deg(\Phi_d(\phi)).$

Proof. Computing:

$$\begin{aligned} \deg(\Phi_n^*(\phi)) &= \sum_{P \in X} \sum_{d|n} \mu\left(\frac{n}{d}\right) i(\Gamma_d, \Delta_X; P) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{P \in X} i(\Gamma_d, \Delta_X; P) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \deg(\Phi_d(\phi)). \end{aligned}$$

□

4.2. Similarities to periodic Lefschetz numbers. Proposition 4.3 looks remarkably similar to the definition of periodic Lefschetz numbers. In this section we describe the connection.

Definition 4.4. Following the notation of [5], define $L(\phi)$ to be the *Lefschetz number* of ϕ . The *periodic Lefschetz number of period n* is then defined as

$$l(\phi^n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) L(\phi^d)$$

The Lefschetz Fixed Point Theorem states that $L(\phi^n) \neq 0$ implies that ϕ^n has a fixed point, in other words, ϕ has a point of period n , but this does not imply that the point is of primitive period n . The periodic Lefschetz numbers were defined to help address this situation. Several papers, including [5, 8], have studied when $l(\phi^n) \neq 0$ implies that there exists a periodic point of primitive period n . We will address the relationship between $\deg(\Phi_n)$, $\deg(\Phi_n^*)$, $L(\phi^n)$, $l(\phi^n)$, and the existence of period points.

Definition 4.5. A map ϕ is *transversal* if $a_P(1) = 1$ for fixed points P .

Proposition 4.6.

- (1) $\deg \Phi_n(\phi) \geq L(\phi^n).$
- (2) If ϕ^n is transversal, then
 - (a) $a_P^*(n) = 1$ if and only if P is a point of primitive period n for ϕ and $a_P^*(n) = 0$ otherwise.
 - (b) $\deg(\Phi_n(\phi))$ is the number of n -periodic points for ϕ .
 - (c) $\deg(\Phi_n^*(\phi))$ is the number of primitive n -periodic points for ϕ . In particular, if $\deg(\Phi_n^*(\phi)) \neq 0$, then there exists a periodic point of primitive period n .

Proof.

- (1) Recall from the Lefschetz-Hopf Theorem that we may compute the Lefschetz number as

$$L(\phi) = \sum_{P \in \text{Fix}(\phi)} \text{ind}(\phi, P)$$

where $\text{ind}(\phi, P)$ is the Poincaré index of ϕ at P . So $L(\phi)$ is the sum of the multiplicities of the fixed points of ϕ with either a negative or positive sign.

- (2) (a) The map ϕ^n is transversal implies that ϕ^d is transversal for all $d | n$ and hence $a_P(d) = 1$ for all periodic points P of period $d | n$. Therefore, if the primitive period of P is n , then we have $a_P^*(n) = a_P(n) = 1$ since $a_P(d) = 0$ for $d < n$. Assume that P is a periodic point of primitive period $m | n$ and compute

$$a_P^*(m) = a_P^*(\phi^m, 1) = a_P(\phi^m, 1) = 1.$$

Since $a_P^*(m) = a_P^*(\phi^m, 1)$, we may replace ϕ by ϕ^m and assume that $m = 1$. Now computing $a_P^*(n)$ we have

$$a_P^*(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_P(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 1 = 0$$

by properties of the Möbius function.

(b) Properties (2b) and (2c) follow directly from the definition of transversal and (2a). \square

Remark. Proposition 4.6(2) is similar to [5, Theorem A].

4.3. Applications.

Proposition 4.7. *There are only finitely many points of primitive period n for any fixed n with ϕ^n non-degenerate.*

Proof. Fix any integer $n \geq 1$ with ϕ^n non-degenerate. Proposition 4.3 provides a formula for the degree of $\Phi_n^*(\phi)$. Since ϕ^n is assumed to be non-degenerate, Bézout's Theorem states that Γ_d and Δ intersect in a finite number of points for all $d \mid n$; in other words, $\deg(\Phi_d(\phi))$ is finite. Hence, $\deg(\Phi_n^*(\phi))$ is finite, so there can only be finitely many primitive n -periodic points. \square

Theorem 4.8. *There exists $M > 0$ such that for all q prime and $q > M$, $\deg(\Phi_q^*(\phi)) \neq 0$ implies that there exists a periodic point with primitive period q for ϕ .*

Proof. We want to show that there exists a P with $a_P^*(q) \neq 0$ that is a primitive q -periodic point. We know that for q prime we have

$$\deg(\Phi_q^*(\phi)) = \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)).$$

There are only finitely many fixed points for ϕ by Proposition 4.7, and for each fixed point only finitely many n relatively prime to the characteristic of K such that $a_P(n) > a_P(1)$ by Theorem 2.20. Hence, after excluding those finitely many numbers (including the characteristic of K), each time $\deg(\Phi_q(\phi)) > \deg(\Phi_1(\phi))$ the additional degree comes from at least one periodic point of primitive period q . \square

Corollary 4.9. *If there are infinitely many $n \in \mathbb{Z}^+$ such that $\deg(\Phi_n^*(\phi)) \neq 0$ for $n \neq 0$ in K and ϕ^n is non-degenerate, then there exists $P \in X$ with an arbitrarily large primitive period for ϕ , and ϕ has infinitely many periodic points.*

Proof. By assumption, we have infinitely many primes q with $\deg(\Phi_q^*(\phi)) \neq 0$. Applying Theorem 4.8, we then have infinitely many primes q with a periodic point of primitive period q . \square

Remark. Corollary 4.9 appears to be similar to applications of periodic Lefschetz numbers such as those in [4, 5].

Theorem 4.10. *If P is a fixed point of ϕ , then the sequence*

$$\{a_P(n)\}_{n \in \mathbb{N}}_{\text{char } K \nmid n}$$

is bounded.

Proof. From Theorem 2.20 we have that for a fixed point P for ϕ , $a_P(n) \neq a_P(1)$ for only finitely many n with $\text{char } K \nmid n$. Hence the sequence must be bounded. \square

Corollary 4.11. *If $\deg(\Phi_n(\phi))$ is unbounded for $\text{char } K \nmid n$, then there are infinitely many periodic points for ϕ and, hence, periodic points with arbitrarily large primitive periods.*

Proof. Consider the prime numbers $q \in \mathbb{Z}$ with $q \neq \text{char } K$. We know that $\deg(\Phi_q(\phi))$ is unbounded, and the only contributions come from fixed points or points of primitive period q . Since the sequence $a_P(q)$ is bounded for all fixed points P , there must be contributions to $\deg(\Phi_q(\phi))$ from periodic points of primitive period q for infinitely many primes q . \square

Remark. Theorem 4.10 and Corollary 4.11 are similar to [17].

4.4. Wehler K3 surfaces. A Wehler K3 surface $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth surface given by the intersection of an effective divisor of degree (1,1) and an effective divisor of degree (2,2). Wehler [20, Theorem 2.9] shows that these surfaces have an infinite automorphism group, from which we have dynamical systems. These dynamical systems were studied in [2, 18].

Theorem 4.12. *Dynamical systems on Wehlers K3 surfaces have points with arbitrarily large primitive period and infinitely many periodic points. In particular, there exists a constant M such that for all primes $q > M$ there exists a periodic point of primitive period q .*

Proof. From [18, page 358] we know that the Lefschetz numbers of the maps $\phi^k = (\sigma_1 \circ \sigma_2)^k$ are given by

$$L(\phi^k) = (2 + \sqrt{3})^{2k} + (2 + \sqrt{3})^{-2k} + 22.$$

So we have

$$(4.1) \quad L(\phi^k) \geq 2^{2k}.$$

By Proposition 4.6(1)

$$(4.2) \quad \deg(\Phi_k(\phi)) \geq L(\phi^k),$$

hence, we have that $\deg(\Phi_k(\phi))$ is unbounded as k increases. Applying Corollary 4.11, we have the result.

To show the second portion, recall that

$$\deg(\Phi_q^*(\phi)) = \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)).$$

In other words, whenever $\deg(\Phi_q(\phi)) > \deg(\Phi_1(\phi))$ we have $\deg(\Phi_q^*(\phi)) \neq 0$. Combining (4.1) and (4.2), we have that for k larger than some constant C we have $\deg(\Phi_q^*(\phi)) \neq 0$. Applying Theorem 4.8 now gives the desired result. \square

Definition 4.13. Let S be a Wehler K3 surface and let \mathcal{A} be the subgroup of the automorphism group of S generated by σ_1 and σ_2 . Let $B_k \subset \mathcal{A}$ be the cyclic subgroup generated by $\phi^k = (\sigma_1 \circ \sigma_2)^k$. Let $\mathcal{A}_P = \{\phi \in \mathcal{A} \mid \phi(P) = P\}$. Let $S[B] = \{P \in S(K) \mid \mathcal{A}_P = B\}$. Recall that we are assuming K is algebraically closed.

The following proposition addresses a remark of Silverman from [18, page 358].

Proposition 4.14. $\#S[B_q] \rightarrow \infty$ as $q \rightarrow \infty$ for q prime.

Proof. From Theorem 4.12 we have that there are periodic points of infinitely large prime primitive period and, in particular, periodic points of prime primitive period for all primes larger than some constant M . Hence, $S[B_q]$ will increase as q increases. \square

4.5. Morphisms of projective space. We also apply our results to morphisms of projective space. Let $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be a morphism of degree d . We need to compute the intersection number for Δ and Γ_{ϕ^n} , which are contained in $\mathbb{P}^N \times \mathbb{P}^N$.

Let D_1 and D_2 be the pullbacks in $\mathbb{P}^N \times \mathbb{P}^N$ of a hyperplane class D in \mathbb{P}^N by the first and second projections, respectively.

Proposition 4.15. *Let Δ and Γ_{ϕ^n} be defined as above.*

(1) The class of Δ is given by

$$\sum_{j=0}^N D_1^{N-j} D_2^j.$$

(2) The class of Γ_{ϕ^n} is given by

$$\sum_{j=0}^N d^{N-j} D_1^{N-j} D_2^j.$$

Proof.

(1) By the Kunneth formula, the diagonal must be a class in

$$H_N(\mathbb{P}^N \times \mathbb{P}^N) = \sum_{j=0}^N H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N).$$

Now, $H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N)$ is a 1-dimensional space for all $0 \leq j \leq N$, spanned by the Poincaré dual of $D_1^{N-j} D_2^j$. We can write

$$\Delta = \sum_{j=0}^N a_j D_1^{N-j} D_2^j.$$

To determine the coefficient a_j , we should intersect Δ with the dual of $D_1^{N-j} D_2^j$. This is $D_1^j D_2^{N-j}$. So let $i_\Delta : \Delta \hookrightarrow \mathbb{P}^N \times \mathbb{P}^N$ and compute

$$(D_1^{N-j} D_2^j) \cdot (\Delta) = i_\Delta^* (D_1^{N-j} D_2^j) \cdot \mathbb{P}^N = D^N \cdot \mathbb{P}^N = 1$$

using the fact that $i_\Delta^*(D_1) = i_\Delta^*(D_2) = D$, a hyperplane class on \mathbb{P}^N .

(2) Again, by the Kunneth formula, the graph must be a class in

$$H_N(\mathbb{P}^N \times \mathbb{P}^N) = \sum_{j=0}^N H_{N-j}(\mathbb{P}^N) \otimes H_j(\mathbb{P}^N),$$

and we can write

$$\Gamma_{\phi^n} = \sum_{j=0}^N a_j D_1^{N-j} D_2^j.$$

Let $i_{\Gamma_n} : \Gamma_n \hookrightarrow \mathbb{P}^N \times \mathbb{P}^N$. To determine the coefficients a_j , we compute

$$(D_1^{N-j} D_2^j) \cdot (\Gamma_{\phi^n}) = i_{\Gamma_n}^* (D_1^{N-j} D_2^j) \cdot \mathbb{P}^N = d^{N-j} D^N \cdot \mathbb{P}^N = d^{N-j}$$

using the facts that $i_{\Gamma_n}^*(D_1) = dD$, since ϕ is degree d , and $i_{\Gamma_n}^*(D_2) = D$.

□

Proposition 4.16. *A morphism $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ of degree d has at most*

$$d^N + d^{N-1} + \cdots + d + 1$$

fixed points.

Proof. By Proposition 4.15, we compute the intersection number of Γ_ϕ and Δ .

$$\begin{aligned} (\Gamma_\phi) \cdot (\Delta) &= \left(\sum_{j=0}^N D_1^{N-j} D_2^j \right) \cdot \left(\sum_{k=0}^N d^{N-k} D_1^{N-k} D_2^k \right) \\ &= 0 + \sum_{j=0}^N d^{N-j} D_1^N D_2^j \\ &= \sum_{j=0}^N d^j. \end{aligned}$$

Since each fixed point has multiplicity at least 1, $\sum_{j=0}^N d^j$ is the maximum possible number of fixed points. \square

Proposition 4.17. *A morphism $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ of degree d has*

$$\deg(\Phi_n(\phi)) = \sum_{j=0}^N (d^n)^j.$$

Proof. ϕ^n has degree d^n so we apply Proposition 4.16 to ϕ^n . \square

Theorem 4.18. *A morphism $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ of degree $d > 1$ has periodic points with arbitrarily large primitive periods and infinitely many periodic points. In particular, there exists a constant M such that for all primes $q > M$ there exist periodic points of primitive period q .*

Proof. The degree $\deg(\Phi_n(\phi))$ is clearly unbounded from Proposition 4.17, so we apply Corollary 4.11 to conclude the first result.

To see the second result, notice that

$$\begin{aligned} \deg(\Phi_q^*(\phi)) &= \deg(\Phi_q(\phi)) - \deg(\Phi_1(\phi)) \\ &= ((d^q)^N + \cdots + (d^q) + 1) - (d^N + \cdots + d + 1) \\ &= ((d^q)^N - d^N) + \cdots + (d^q - d) \\ &> 0. \end{aligned}$$

Hence, we apply Theorem 4.8 to conclude the result. \square

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